STABILITY OF STOCHASTIC DYNAMIC EQUATIONS WITH TIME-VARYING DELAY ON TIME SCALES

Le Anh Tuan

Ha noi University of industrial, Vietnam

Busan- 11/2017

The aim of talk

This talk presents some our recent results related to the Stability of stochastic dynamic equations with time-varying delay on time scales. We divide this talk into three main parts

1. Introduce
2. Preliminaries
3. ∇-Stochastic dynamic delay equations
4. Exponential p-stability of stochastic dynamic delay equations
5. Almost sure exponential stability of dynamic delay equations

Introduce

- In recent years, the theory of the analysis on the time scale, which was introduced by S. Hilger in his PhD thesis, has been born in order to unify continuous and discrete analysis.
- As far as we know, there are very few works dealing the dynamic delay equations on time scales. The main reason is that the subtraction on a time scale, in general, is no longer valid, which causes the difficulty to drive a concept of "delay equations on time scales". In X. L. Liu, W. X. Wang, J. Wu (2010) and Y. Ma, J. Sun (2007), the authors try to give the concept of delay equation and consider qualitative properties of solutions of deterministic dynamic delay equation on time scales. However, the assumptions imposed on time scales in this paper are too strict.

Introduce

- Therefore, in this topic we try to give a definition for stochastic dynamic delay equation, which is more available, and consider the existence and uniqueness of solutions. After that, we use Lyapunov function to give sufficient conditions for the uniformly exponentially p-stable, exponential almost sure stability.
- Since the substitution rule in integral can not apply in the calculus on time scale, this study is not a simple unification of some known results on the difference/differential delay equations. To obtain these results, we have to use some new techniques in the proof of theorems.

Preliminaries and Definitions on times

- A time scale is a nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ is endowed with the topology inherited from the real numbers with the standard topology.
- Let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \mu(t) = \sigma(t) - t$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \nu(t) = t - \rho(t)$ (supplemented by $\sup \emptyset = \inf \mathbb{T}, \inf \emptyset = \sup \mathbb{T}$). A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered.
- If $\mathbb{T}$ has a right-scattered minimum $M_{\min}$, then define ${}_k\mathbb{T} = \mathbb{T} \setminus \{M_{\min}\}$, otherwise ${}_k\mathbb{T} = \mathbb{T}$.

Preliminaries and Definitions on times

- A function f defined on $\mathbb{T}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point. A regulated function is called *ld-continuous* if it is continuous at every left-dense point. Similarly, one has the notion of *rd*-continuous. Denote $\mathbb{T}_a = \{t \in \mathbb{T} : t \geq a\}$ and by $\mathcal{R}$ (*resp.* $\mathcal{R}^+$)the set of all *rd*-continuous and regressive (resp. positive regressive) functions.
- For any function f defined on $\mathbb{T}$, we write $f_t^\rho = f(\rho(t))$ for all $t \in {}_k\mathbb{T}$ and $\lim_{\sigma(s)\uparrow t} f(s)$ by $f(t_-)$ if this limit exists. It is easy to see that if t is left-scattered then $f_{t_-} = f_t^\rho$. Throughout of this topic, we suppose that the time scale $\mathbb{T}$ has bounded graininess, that is $\nu_* = \sup\{\nu(t) : t \in {}_k\mathbb{T}\} < \infty$.

Preliminaries and Definitions on times

- On time scales, we have achieved basic results about: ∇- derivatives and ∇-integral of a function defined on time scales; stochastic process, predictable process, martingale, semimartingale, stopping time indexed by a time scale; Doob-Meyer expansion; ∇-stochastic integration on time scales; Itô's formula and applying the Itô's formula to the martingale problem (If the reader is interested in it, these notions can be found in many documents and papers, such as M. Bohner and A. Peterson (2001), N.H. Du and N.T. Dieu (2011, 2013)...).

Preliminaries and Definitions on times

- if $p \in \mathcal{R}$ then the exponential function $e_p(t, t_0)$, is solution of the initial value problem

$$y^{\nabla}(t) = p(t_-)y(t_-), \;\; y(t_0) = 1, \;\; t > t_0. \tag{2.1}$$

Also if $p \in \mathcal{R}$, $e_{\ominus p}(t, t_0)$ is the solution of the equation

$$y^{\nabla}(t) = -p(t_-)y(t), \;\; y(t_0) = 1, \;\; t > t_0, \tag{2.2}$$

where $\ominus p(t) = \frac{-p(t)}{1+\mu(t)p(t)}$.

Preliminaries and Definitions on times

Later, we need the following lemma:

Lemma 2.1

Let $u(t)$ be a regulated function and $u_a, \alpha \in \mathbb{R}_+$. Then, the inequality

$$u(t) \leq u_a + \alpha \int_a^t u(\tau_-)\nabla\tau \quad \textit{for all} \quad t \in \mathbb{T}_a$$

implies

$$u(t) \leq u_a e_\alpha(t, a) \quad \textit{for all} \;\; t \in \mathbb{T}_a.$$

3.1. ∇-Stochastic delay dynamic equations

- Let $\mathbb{T}$ be a time scale. We say that the *ld*-continuous map $r(\cdot) : {}_k\mathbb{T} \to \mathbb{T}$ is a delay function if $r(t) \leq t_-$ for all $t \in \mathbb{T}$ and $r_* = \sup\{t - r(t) : t \in \mathbb{T}\} < \infty$. For any $s \in \mathbb{T}$, we see that $b_s := \min\{r(t) : t \geq s\} > -\infty$. Denote $\Gamma_s = \{r(t) : t \geq s\} \cap [b_s, s]$ and by $C(\Gamma_s;\ \mathbb{R}^d)$ the family of continuous functions from Γ_s to $\mathbb{R}^d$ with the norm $\|\varphi\|_s = \sup_{s\in\Gamma_s} \|\varphi(s)\|$.
- Fix $t_0 \in \mathbb{T}$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}}$ satisfying the usual conditions (i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}}$ is increasing and right continuous while $\mathcal{F}_{t_0}$ contains all P-null sets). Denote by $\mathcal{M}_2$ the set of the square integrable martingales defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}}, \mathbb{P})$ and by $\mathcal{M}_2^r$ the subspace of the space $\mathcal{M}_2$ consisting of martingales with continuous characteristics.Throughout of this paper, we fix a $M = \{M_t\}_{t\geq t_0} \in \mathcal{M}_2$ with the characteristic $\langle M\rangle_t$.

3.1. ∇-Stochastic dynamic delay equations

- Suppose that $\langle M\rangle_t$ is absolutely continuous with respect to Lebesgue measure μ_∇, i.e., there exists $\mathcal{F}_t$-adapted progressively measurable process K_t such that

$$\langle M\rangle_t = \int_{t_0}^{t} K_\tau \nabla\tau. \tag{3.1}$$

Further, for any $T \in \mathbb{T}_{t_0}$, there is a constant N (possibly depending on T) such that

$$\mathbb{P}\{\mathrm{esssup}_{t_0\le t\le T}|K_t| \le N\} = 1. \tag{3.2}$$

Denote by $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{T}_{t_0};\mathbb{R}^d)$ (resp. $\mathcal{L}_2^{\mathrm{loc}}(\mathbb{T}_{t_0};\mathbb{R}^d, M)$) the set of functions, valued in $\mathbb{R}^d$, $\mathcal{F}_t$-adapted such that $\int_{t_0}^{T} f(t)\nabla t < \infty$, (resp. $\mathbb{E}\int_{t_0}^{T} h^2(t)\nabla\langle M\rangle_t < \infty$) $\forall\, T \in \mathbb{T}_{t_0}$.

3.1. ∇-Stochastic delay dynamic equations

Let $r(t)$ be a delay function. We now consider the ∇-stochastic dynamic delay equations on time scale

$$\begin{cases} d^{\nabla}X(t) = f(t, X(t_-), X(r(t)))d^{\nabla}t + g(t, X(t_-), X(r(t)))d^{\nabla}M_t \\ X(s) = \xi(s) \ \ \forall\, s \in \Gamma_{t_0}, t \in \mathbb{T}_{t_0}, \end{cases} \tag{3.3}$$

where $f : \mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$; $g : \mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are two Borel functions and $\xi = \{\xi(s) : \ s \in \Gamma_{t_0}\}$ is a $C(\Gamma_{t_0}; \mathbb{R}^d)$-valued, $\mathcal{F}_{t_0}$-measurable random variable with $E\|\xi\|^2_{t_0} < +\infty$. In the following we denote by $\widetilde{\mathbb{T}}_s$ the set $\Gamma_s \cup \mathbb{T}_s$ for any $s \in \mathbb{T}$.

3.1. ∇-Stochastic delay dynamic equations

Definition 3.1

A stochastic process $(X(t))_{t\in\widetilde{\mathbb{T}}_{t_0}}$, valued in $\mathbb{R}^d$, is called the solution of the equation (3.3) if

(i) $\{X(t)\}$ is $\{\mathcal{F}_t\}$-adapted;

(ii) $f(\cdot, X(\cdot_-), X(r(\cdot))) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{T}_{t_0}; \mathbb{R}^d)$;

(iii) $g(\cdot, X(\cdot_-), X(r(\cdot))) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{T}_{t_0}; \mathbb{R}^d, M)$;

(iv) $X(t) = \xi(t)\ \forall\, t \in \Gamma_{t_0}$ and for any $t \in \mathbb{T}_{t_0}$ and with probability 1 there holds the equation

$$X(t) = \xi(t_0) + \int_{t_0}^{t} f(s, X(s_-), X(r(s)))\nabla s + \int_{t_0}^{t} g(s, X(s_-), X(r(s)))\nabla M_s,\ \forall\, t \in \mathbb{T}_{t_0}. \quad (3.4)$$

3.1. ∇-Stochastic delay dynamic equations

- The equation (3.3) is said to have the uniqueness of solutions if $X(t)$ and $\overline{X}(t)$ with $X(t) = \overline{X}(t)$ for $t \in \Gamma_{t_0}$ are two processes satisfying (3.4) then
$$P\{X(t) = \overline{X}(t) \, \forall \, t \in \mathbb{T}_{t_0}\} = 1.$$
- It is seen that $\int_{t_0}^{t} g(s, X(s_-), X(r(s)))\nabla M_s$ is $\mathcal{F}_t$-martingale so it has a cadlag modification. Hence, if $X(t)$ satisfies (3.4) then $X(t)$ is cadlag. In addition, if M_t is *rd*-continuous, so is $X(t)$.
- We now give conditions guaranteeing the existence and uniqueness of solutions to Equation (3.3). Firstly, we consider the case where coefficients satisfy Lipschitz and Sub-linear growth rate conditions.

3.2. Existence and uniqueness of solutions

Theorem 3.2

Assume that for any $T \in \mathbb{T}_{t_0}$, there exist two positive constants $\kappa = \kappa(T)$ and $\overline{\kappa} = \overline{\kappa}(T)$ such that
(i) (Lipschitz condition) for all $x_i, y_i \in \mathbb{R}^d, \quad i = 1, 2,$ and $t \in [t_0, T]$

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\|^2 \vee \|g(t, x_1, y_1) - g(t, x_2, y_2)\|^2 \leq \overline{\kappa}(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2). \quad (3.5)$$

(ii) (Linear growth condition) for all $(t, x, y) \in [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d$

$$\|f(t, x, y)\|^2 \vee \|g(t, x, y)\|^2 \leq \kappa(1 + \|x\|^2 + \|y\|^2). \quad (3.6)$$

Then, there exists a unique solution $X(t)$ to Equation (3.3) and this solution is a square integrable semimartingale.

3.3. Rate of the convergence

Theorem 3.3

Let the assumptions of Theorem 3.2 hold. Let $X(t)$ be the unique solution of Equation (3.3) and $X_n(t)$ be the Picard iteration defined by:

$$X_n(t) = \xi(t) \, \forall \, t \in \Gamma_{t_0};$$

and

$$X_n(t) = \xi(t_0) + \int_{t_0}^{t} f\Big(s, X_{n-1}(s_-), X_{n-1}(r(s))\Big)\nabla s + \int_{t_0}^{t} g\Big(s, X_{n-1}(s_-), X_{n-1}(r(s))\Big)\nabla M_s, t \geq t_0 \quad (3.7)$$

. *with* $n = 1, 2, \ldots.$

3.3. Rate of the convergence

Then,

$$\mathbb{E}\Big(\sup_{t_0\le t\le T}\|X_n(t)-X(t)\|^2\Big)\le 2Ce_{2\overline{P}}(T,t_0)\overline{P}^n h_n(T,t_0), \tag{3.8}$$

for all $n\ge 1$, where C and $\overline{P}$ are defined in the proof of Theorem 3.2, i.e., $C=2\kappa\left[(T-t_0)^2+4N(T-t_0)\right](1+2\mathbb{E}\|\xi\|_{t_0}^2)$; $\overline{P}=4\overline{\kappa}(T-t_0+4N)$.

3.4. Condition locally Lipschitz on existence and uniqueness of solutions

a. Infinitesimal operator

Let $C^{1,2}([a,b]\times\mathbb{R}^d;\mathbb{R})$ be the set of all functions $V(t,x)$ defined on $[a,b]\times\mathbb{R}^d$, having continuous ∇-derivative in t and continuous second derivative in x. For any $V\in C^{1,2}(\mathbb{T}_{t_0}\times\mathbb{R}^d;\mathbb{R})$ define

$$
\begin{aligned}
&\mathcal{L}V(t,x,y) = V^{\nabla_t}(t,x)\\
&+\sum_{i=1}^{d}\frac{\partial V(t,x)}{\partial x_i}(1-1_{\mathbb{I}}(t))f_i(t,x,y)+\Big(V(t,x+f(t,x,y)\nu(t))-V(t,x)\Big)\Phi(t)\\
&+\frac{1}{2}\sum_{i,j}\frac{\partial^2 V(t,x)}{\partial x_i x_j}g_i(t,x,y)g_j(t,x,y)\widehat{K}_t^c-\sum_{i=1}^{d}\frac{\partial V(t,x)}{\partial x_i}g_i(t,x,y)\int_{\mathbb{R}}u\widehat{\Upsilon}(t,du\\
&+\int_{\mathbb{R}}(V\big(t,x+f(t,x,y)\nu(t)+g(t,x,y)u\big)-V(t,x+f(t,x,y)\nu(t)))\Upsilon(t,du),
\end{aligned}
\tag{3.9}
$$

3.4. Condition locally Lipschitz on existence and uniqueness of solutions

where V^{∇_t} is partial ∇-derivative of $V(t, x)$ in t and

$$\Phi(t) = \begin{cases} 0 & \text{if } t \text{ left-dense} \\ \frac{1}{\nu(t)} & \text{if } t \text{ left-scattered.} \end{cases}$$

Set

$$H_t = V(t, X(t)) - V(t_0, X(t_0)) - \int_{t_0}^{t} \mathcal{L}V(s, X(s_-), X(r(s)))\nabla s. \quad (3.10)$$

By using the Itô's formula ([2, Theorem 1, pp.322]) we see that $(H_t, \mathcal{F}_t)_{t\in\mathbb{T}_{t_0}}$ is a locally integrable martingale.

3.4. Condition locally Lipschitz on existence and uniqueness of solutions

b. Condition locally Lipschitz

Theorem 3.4

Suppose that for any $k > 0$ *and* $T \in \mathbb{T}_{t_0}$ *there exists a constant* $L_{T,k} > 0$*:*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\|^2 \vee \|g(t,x_1,y_1) - g(t,x_2,y_2)\|^2 \leq L_{T,k}(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2), \text{for all } x_i, y_i \in \mathbb{R}^d, \quad i = 1,2, \tag{3.11}$$

with $\|x_i\| \vee \|y_i\| \leq k$ *and* $t \in \mathbb{T}_{t_0}$*. Further, there are two positive constants* λ_1, λ_2 *and a function* $V \in C^{1,2}([b_{t_0}, T] \times \mathbb{R}^d; \mathbb{R}_+)$ *satisfying*

$$\mathcal{L}V(t,x,y) \leq \lambda_1 V(t_-, x) + \lambda_2 V(r(t), y), \tag{3.12}$$

and $\lim\limits_{\|x\|\to\infty} \inf\limits_{t\in[t_0,T]} V(t,x) = \infty$*. Then, the equation* (3.3) *has a unique solution* $X(t)$ *defined on* $\widetilde{\mathbb{T}}_{t_0}$*.*

4.1. Basic definition

We suppose that for any $s > t_0$ and $\xi \in C(\Gamma_s;\ \mathbb{R}^d)$, there exists a unique solution $X(t, s, \xi), t \in \widetilde{\mathbb{T}}_s$ of the equation (3.3) satisfying $X(t, s, \xi) = \xi(t)$ for any $t \in \Gamma_s$. Further,

$$f(t, 0, 0) \equiv 0;\ g(t, 0, 0) \equiv 0, \ \ \forall\, t \in \mathbb{T}_{t_0}. \tag{4.1}$$

From the condition (4.1), Equation (3.3) has a trivial solution $X(t, s, 0) \equiv 0$.

4.1. Basic definitions

Definition 4.1

The trivial solution $X(t, s, 0) \equiv 0$ of the equation (3.3) is said to be exponentially p-stable if there is a positive constant α such that for any $s > t_0$, there exists $\beta_s > 0$ for which the following relation

$$\mathbb{E}\|X(t, s, \xi)\|^p \leq \beta_s \|\xi\|_s^p e_{\ominus\alpha}(t, s) \text{ on } t \geq s, \tag{4.2}$$

holds for any $\xi \in C(\Gamma_s;\ \mathbb{R}^d)$.

• If one can choose β_s independent of s, the trivial solution of the equation (3.3) is called uniformly exponentially p-stable. When $p = 2$, it is usually said to be exponentially stable in mean square.

4.2. Sufficient conditions for the exponential p-stability

Theorem 4.2

Let $\alpha_1, \alpha_2, p, c_1, c_2$ be positive numbers with $\alpha_1 > \alpha_2$. Suppose that there exists a positive definite function $V \in C^{1,2}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R}_+)$ such that

$$c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p \quad \forall (t,x) \in \mathbb{T} \times \mathbb{R}^d, \tag{4.3}$$

and for all $(t,x,y) \in \mathbb{T}_{t_0} \times \mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{L}V(t,x,y) \le -\frac{\alpha_1}{1+\alpha_1\nu(t)} V(t_-,x) + \frac{\alpha_2 e_{\ominus\alpha_1}(t_-, r(t))}{1+\alpha_2\nu(t)} V(r(t),y). \tag{4.4}$$

Then, the equation (3.3) is uniformly exponentially p-stable.

4.3. Examples

• We now consider a special case. Let P be a positive definite matrix and $V(t,x) = x^\top Px$, where $x^\top$ is the transpose of a vector x. By using (3.9) and by directly calculating we obtain

$$\begin{aligned} \mathcal{L}V(t,x,y) &= x^\top Pf(t,x,y) + f(t,x,y)^\top Px + f(t,x,y)^\top Pf(t,x,y)\nu(t) \\ &\quad + g(t,x,y)^\top Pg(t,x,y)K_t. \end{aligned} \tag{4.5}$$

4.3. Examples

Example 4.3.1

• Let $\mathbb{T}$ be a time scale containing 0 and $r(t)$ be a delay function. Consider the stochastic dynamic delay equation on time scale $\mathbb{T}$

$$\begin{cases} d^\nabla X(t) = AX(t_-)d^\nabla t + BX(r(t))d^\nabla W(t) \\ X(s) = \xi(s) \ \ \forall\, s \in \Gamma_0, t \in \mathbb{T}_0, \end{cases} \tag{4.6}$$

where A and B are $d \times d$ matrices. Let $V(t, x) = \|x\|^2$.

Suppose that the spectral abscissa of the matrix $A + A^\top + A^\top A\nu(t)$ is uniformly bounded by a negative constant $-\alpha_1$ and there exists a positive constant α_2 such that $\alpha_2 < \alpha_1$ and $\|B\|^2 e^{r_* \alpha_1} \le \frac{\alpha_2}{1+\nu^* \alpha_2}$. For this assumption we obtain

$$\mathcal{L}V(t,x,y) \le -\frac{\alpha_1}{1+\alpha_1\nu(t)}\|x\|^2 + \frac{\alpha_2 e_{\ominus\alpha_1}(t_-, r(t))}{1+\alpha_2\nu(t)}\|y\|^2.$$

Therefore, assumptions of Theorem 5.2 are satisfied with $p = 2$, it means the trivial solution of Equation (4.6) is exponentially stable in mean square.

4.3. Examples

Example 4.3.2

Let $\mathbb{T}$ be a time scale defined by

$$\mathbb{T} = \mathbb{P}_{\frac{1}{4},1} = \bigcup_{k=1}^{\infty} \left[\frac{5k}{4}, \frac{5k+4}{4}\right].$$

Let $r(t)$ be a delay function satisfying $r_* = \sup_{t\in\mathbb{T}}(t - r(t)) = \frac{1}{4}$. Consider the stochastic dynamic delay equation on time scale $\mathbb{T}$

$$\begin{cases} d^\nabla X(t) = \big(AX(t_-) + \frac{1}{2}X(r(t))\big)d^\nabla t + BX(t_-)d^\nabla W(t), t \geq t_0 \\ X(s) = \xi(s) \ \ \forall\, s \in \Gamma_{t_0}, \end{cases} \tag{4.7}$$

where A, B are the 3×3 matrices defined by

$$A = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -2 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{7}{3} \end{bmatrix}; B = \begin{bmatrix} \frac{11}{18} & -\frac{2}{9} & \frac{5}{18} \\ -\frac{2}{9} & \frac{4}{9} & -\frac{1}{18} \\ \frac{5}{18} & -\frac{1}{18} & \frac{25}{36} \end{bmatrix}.$$

With the Lyapunov function $V(t,x) = \|x\|^2$, by direct calculation we have

$$\mathcal{L}V(t,x,y) \le -\frac{143}{144}\|x\|^2 + \frac{1}{2}\|y\|^2. \tag{4.8}$$

Setting $\alpha_1 := \frac{143}{144}$, $\alpha_2 := \frac{4}{5}$ then α_1, α_2 satisfy the inequalities $\frac{1}{2}e^{\alpha_1 r_*} < \frac{\alpha_2}{1+\nu_*\alpha_2}$. Combining these estimations and (4.8), we obtain

$$\begin{aligned}\mathcal{L}V(t,x,y) &\le -\alpha_1\|x\|^2 + \frac{\alpha_2}{1+\nu_*\alpha_2}e^{-\alpha_1 r_*}\|y\|^2 \\ &\le -\frac{\alpha_1}{1+\alpha_1\nu(t)}\|x\|^2 + \frac{\alpha_2 e_{\ominus\alpha_1}(t_-, r(t))}{1+\alpha_2\nu(t)}\|y\|^2.\end{aligned}$$

By virtue of Theorem 5.2 the trivial solution of Equation (4.7) is exponentially stable in mean square.

5.1. Basic definition

Definition 5.1

The trivial solution $X(t) \equiv 0$ of the equation (3.3) is said to be almost surely exponentially stable if for any $s \in \mathbb{T}_{t_0}$ the relation

$$\limsup_{t\to\infty} \frac{\log \|X(t,s,\xi)\|}{t} < 0 \tag{5.1}$$

holds for any $\xi \in C(\Gamma_s;\ \mathbb{R}^d)$.

5.2. Sufficient conditions for the almost sure exponential stability

Theorem 5.2

Let $\alpha_1, \alpha_2, p, c_1$ be positive numbers with $\alpha_1 > \alpha_2$. Let α be a positive number satisfying $\frac{\alpha}{1+\alpha\nu(t)} < \alpha_1$ and let η be a non-negative ld-continuous function defined on $\mathbb{T}_{t_0}$ such that $\int_{t_0}^{\infty} e_\alpha(\tau_-, t_0)\eta_t \nabla t < \infty$ a.s.. Suppose that there exists a positive definite function $V \in C^{1,2}(\mathbb{T}_{t_0} \times \mathbb{R}^d; \mathbb{R}_+)$ satisfying $c_1\|x\|^p \le V(t,x) \quad \forall (t,x) \in \mathbb{T}_{t_0} \times \mathbb{R}^d$, and for all $t \ge t_0$,

$$V^{\nabla_t}(t,x) + \mathcal{A}V(t,x,y) \le -\alpha_1 V(t_-,x) + \eta_t \qquad a.s., \tag{5.2}$$

for all $x \in \mathbb{R}^d$ and $t \ge t_0$. Then, the trivial solution of equation (3.3) is almost surely exponentially stable.

References

- M. Bohner, O. M. Stanzhytskyi and A. O. Bratochkina, Stochastic dynamic equations on general time scales, *Electronic Journal of Differential Equations*, 2013(57)(2013) 1-15.
- N.H. Du and N.T. Dieu, Stochastic dynamic equations on time scales, *Acta Mathematica Vietnamica*, vol.38, n0 2, DOI: 10.1007/s40306-013-0022-3 2013.
- D. Grow and S. Sanyal. Brownian motion indexed by a time scale. *Stochastic Analysis and Applications*, 29(2011) 457 - 472.
- A. L. Liu, Boundedness and exponential stability of solutions to dynamic equations on time scales, *Electron. J. Diff. Equ.*, 2007, no. 12, 1-14.
- X. L. Liu, W. X. Wang and J. Wu (2010), Dynamic delay equations on time scales, *Appl. Anal.*, **8**, 1241-1249.
- X. L. Liu (2007), Boundedness and exponential stability of solutions to dynamic equations on time scales, *Electron. J. Diff. Eq.*, **12**, 1-14

Thank you for your attention!